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More on coupling coefficients for the most degenerate representations of $SO(n)$

Markus Hormeß and Georg Junker

Institut für Theoretische Physik, Universität Erlangen-Nürnberg, Staudtstraße 7, D-91058 Erlangen, Germany

E-mail: hormess@theorie1.physik.uni-erlangen.de and junker@theorie1.physik.uni-erlangen.de

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Abstract. We present explicit closed-form expressions for the general group-theoretical factor appearing in the α-topology of a high-temperature expansion of $SO(n)$-symmetric lattice models. This object, which is closely related to $6j$-symbols for the most degenerate representation of $SO(n)$, is discussed in detail.

1. Introduction

In this paper we extend our previous studies [1] on coupling coefficients for the so-called most degenerate (also called symmetric or class-one) representations of $SO(n)$. These coupling coefficients are important in many fields of theoretical physics such as atomic and nuclear physics. For example, in connection with the Jahn–Teller effect an extensive study of particular $6j$-symbols is due to Judd and co-workers [2]. A detailed study of isoscalar factors of $SO(n) \supset SO(n-1)$ and related $6j$-coefficients has been made by Ališauskas [3], showing that the $6j$-coefficients of $SO(n)$ can be expressed in terms of (generalized) $6j$-coefficients of $SU(2)$.

Coupling coefficients of the most degenerate representations of $SO(n)$ also appear as group-theoretical factors in the high-temperature expansion of $SO(n)$-symmetric classical lattice models [4, 5] such as the XY-model ($n = 2$) and the Heisenberg model ($n = 3$). In this paper we present new explicit results for the so-called α-graph, which contributes with the following group-theoretical factor to the high-temperature expansion of the free energy of such models [1, 4, 5]:

$$\begin{aligned} I_n &\equiv I_n(\ell_1, \ell_2, \ell_3|\ell_4, \ell_5, \ell_6) \\ &:= \int_{SO(n)} dg_1 \int_{SO(n)} dg_2 \int_{SO(n)} dg_3\, \mathcal{D}^{\ell_1}_{00}(g_1)\, \mathcal{D}^{\ell_2}_{00}(g_2)\, \mathcal{D}^{\ell_3}_{00}(g_3)\, \mathcal{D}^{\ell_4}_{00}(g_2^{-1}g_3) \\ &\quad \times \mathcal{D}^{\ell_5}_{00}(g_3^{-1}g_1)\, \mathcal{D}^{\ell_6}_{00}(g_1^{-1}g_2). \end{aligned} \tag{1}$$

Here $\mathcal{D}^{\ell}_{00}(g)$ denotes a particular matrix element (the zonal spherical function) of the ℓth unitary irreducible class-one representation of $SO(n)$, $\ell \in \mathbb{N}_0$, and dg is the normalized invariant Haar measure on $SO(n)$. For details we refer to our earlier work [1]. Here we only note the relation of the above integral with the $6j$-symbols of $SO(n)$:

$$I_n = (-1)^{\ell_4+\ell_5+\ell_6} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix}_{(n)} \begin{pmatrix} \ell_1 & \ell_5 & \ell_6 \\ 0 & 0 & 0 \end{pmatrix}_{(n)} \begin{pmatrix} \ell_4 & \ell_2 & \ell_6 \\ 0 & 0 & 0 \end{pmatrix}_{(n)} \times \begin{pmatrix} \ell_3 & \ell_4 & \ell_5 \\ 0 & 0 & 0 \end{pmatrix}_{(n)} \begin{Bmatrix} \ell_1 & \ell_2 & \ell_3 \\ \ell_4 & \ell_5 & \ell_6 \end{Bmatrix}_{(n)} \quad (2)$$

where

$$\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix}_{(n)}^2 := \int_{SO(n)} \mathrm{d}g\, \mathcal{D}_{00}^{\ell_1}(g)\, \mathcal{D}_{00}^{\ell_2}(g)\, \mathcal{D}_{00}^{\ell_3}(g) = \frac{(J+n-3)!}{(n-3)!\,\Gamma^2(n/2)\,\Gamma(J+n/2)} \prod_{i=1}^{3} \left[\frac{(n-2)!\,\ell_i!\,\Gamma(J-\ell_i+(n-2)/2)}{2\,(\ell_i+n-3)!\,(J-\ell_i)!} \right] \quad (3)$$

denotes the square of a $3j$-symbol, which vanishes unless $J := (\ell_1+\ell_2+\ell_3)/2$ is a non-negative integer, $J \in \mathbb{N}_0$, and the ℓ's obey the triangular relation well known from the case $n = 3$. This result, in essence, goes back to an earlier one of Vilenkin [6] (equation (6), p 490; see also the work of Ališauskas [7] and references therein). A derivation of (3) can be found in [1], equations (21)–(24), where a phase convention for the $3j$-symbol is also given. This together with an explicit expression for I_n then leads to a closed-form expression for the $6j$-symbol, which is denoted with curly brackets in (2). The resulting expressions are indeed similar to those obtained by Ališauskas [3].

The purpose of this paper is to derive a rather elementary expression for the above group integral $I_n(\ell_1, \ell_2, \ell_3|\ell_4, \ell_5, \ell_6)$ which allows us to present, for given but arbitrary values of the ℓ's and any n, explicit results for (1). So far only particular results have been given in the literature. For example, for arbitrary n and $(\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6) = (1, 1, 2, 1, 1, 2)$ an explicit expression has been given by Domb [5], the elementary case $\ell_4 = 0$ can be found in [1]† and, rather recently, some results have been given for the cases where one of the ℓ's equals one or two [9].

The remaining part of this paper deals with the derivation of an elementary expression for I_n, which is given below in (10) in combination with (6), (14) and (15). Together with the above expression for the $3j$-symbol we have thus also obtained a new elementary expression for the corresponding $6j$-symbol. We finally present some explicit results for arbitrary n and $\ell_i \in \{1, 2, 3, 4\}$ after briefly discussing the symmetry properties of I_n. Our result will also be compared with that of Ališauskas [3].

2. Explicit integration of (1)

In this section we will make extensive use of our previous results [1]. In the following if we refer to equations of [1] we will add the superscript 1 to the equation number. For example, $(18)^1$ refers to equation (18) of [1] which shows that the zonal spherical functions can be expressed in terms of Gegenbauer polynomials. In fact, using this relation the integral (1) may

† Note that equation (47) in [1] should read $\begin{Bmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & \ell_5 & \ell_6 \end{Bmatrix}_{(n)} = ((-1)^{\ell_1+\ell_2+\ell_3}/\sqrt{d_{\ell_2}d_{\ell_3}})\,\delta_{\ell_2\ell_6}\delta_{\ell_3\ell_5}$ if ℓ_1, ℓ_2, ℓ_3 obey the triangular condition and vanishes otherwise.

be rewritten as follows:

$$I_n = \left[\prod_{i=1}^{6} \frac{\ell_i!\,(n-3)!}{(\ell_i+n-3)!}\right] \int_{S^{n-1}} \frac{\mathrm{d}^{n-1}e_1}{|S^{n-1}|} \int_{S^{n-1}} \frac{\mathrm{d}^{n-1}e_2}{|S^{n-1}|} \int_{S^{n-1}} \frac{\mathrm{d}^{n-1}e_3}{|S^{n-1}|}$$
$$\times C_{\ell_1}^{(n-2)/2}(a\cdot e_1)\, C_{\ell_2}^{(n-2)/2}(a\cdot e_2)\, C_{\ell_3}^{(n-2)/2}(a\cdot e_3)$$
$$\times C_{\ell_4}^{(n-2)/2}(e_2\cdot e_3)\, C_{\ell_5}^{(n-2)/2}(e_3\cdot e_1)\, C_{\ell_6}^{(n-2)/2}(e_1\cdot e_2). \tag{4}$$

Here and in the following we will use the same notation as in [1]. Denoting by θ_i the polar angle of the unit vector $e_i \in S^{n-1}$ we have $e_i = (\sin\theta_i f_i, \cos\theta_i)$ with $f_i \in S^{n-2}$. Using $a \cdot e_i = \cos\theta_i$ and the addition theorem for Gegenbauer polynomials [8]

$$C_\ell^{n/2-1}(e_i\cdot e_j) = \sum_{m=0}^{\ell} a(n,\ell,m)\, \sin^m\theta_i\, C_{\ell-m}^{m+n/2-1}(\cos\theta_i)\, \sin^m\theta_j\, C_{\ell-m}^{m+n/2-1}(\cos\theta_j)$$
$$\times C_m^{(n-3)/2}(f_i\cdot f_j) \tag{5}$$

where we have set

$$a(n,\ell,m) := \frac{2^{2m}(n-4)!\,(\ell-m)!\,\Gamma^2(m+n/2-1)}{(\ell+m+n-3)!\,\Gamma^2(n/2-1)}\,(2m+n-3) \tag{6}$$

the above integrations can be factorized into those over the polar angles and the remaining integrals over S^{n-2}. For this we have also to make use of (39)[1] in the form

$$\int_{S^{n-1}} \frac{\mathrm{d}^{n-1}e}{|S^{n-1}|}\,(\cdot) = \frac{\Gamma(n/2)}{\sqrt{\pi}\,\Gamma((n-1)/2)} \int_0^\pi \mathrm{d}\theta\, \sin^{n-2}\theta \int_{S^{n-2}} \frac{\mathrm{d}^{n-2}f}{|S^{n-2}|}\,(\cdot). \tag{7}$$

The part of (4) which involves the f-integrations reads ($m_i = 0, 1, \ldots, \ell_{3+i}$)

$$F_n := \int_{S^{n-2}} \frac{\mathrm{d}^{n-2}f_1}{|S^{n-2}|} \int_{S^{n-2}} \frac{\mathrm{d}^{n-2}f_2}{|S^{n-2}|} \int_{S^{n-2}} \frac{\mathrm{d}^{n-2}f_3}{|S^{n-2}|}$$
$$\times\, C_{m_1}^{(n-3)/2}(f_2\cdot f_3)\, C_{m_2}^{(n-3)/2}(f_3\cdot f_1)\, C_{m_3}^{(n-3)/2}(f_1\cdot f_2)$$
$$= \prod_{i=1}^{3}\left[\frac{(m_i+n-4)!}{m_i!\,(n-4)!}\right]$$
$$\times \int_{SO(n-1)} \mathrm{d}h_1 \int_{SO(n-1)} \mathrm{d}h_2 \int_{SO(n-1)} \mathrm{d}h_3\, D_{00}^{m_1}(h_2^{-1}h_3)\, D_{00}^{m_2}(h_3^{-1}h_1)\, D_{00}^{m_3}(h_1^{-1}h_2) \tag{8}$$

where D_{00}^{m} denotes zonal spherical functions of the subgroup $SO(n-1)$. These group integrations are easily performed via the orthogonality relation for the $SO(n-1)$ matrix elements D_{00}^m, cf (12)[1]. Consequently, all m_i's have to be equal, $m \equiv m_1 = m_2 = m_3 = 0, 1, 2, \ldots, \min\{\ell_4, \ell_5, \ell_6\}$, and the result reads

$$F_n = \sum_{m=0}^{\min\{\ell_4,\ell_5,\ell_6\}} \delta_{mm_1}\delta_{mm_2}\delta_{mm_3}\, \frac{(m+n-4)!}{m!\,(n-4)!}\left(\frac{n-3}{2m+n-3}\right)^2. \tag{9}$$

With the help of this result we are now able to put the quantity of our interest into the form

$$I_n = \left[\prod_{i=1}^{6} \frac{\ell_i!\,(n-3)!}{(\ell_i+n-3)!}\right]\left(\frac{\Gamma(n/2)}{\sqrt{\pi}\,\Gamma((n-1)/2)}\right)^3$$
$$\times \sum_{m=0}^{\min\{\ell_4,\ell_5,\ell_6\}} \frac{(m+n-4)!}{m!\,(n-4)!}\left(\frac{n-3}{2m+n-3}\right)^2 \left[\prod_{i=4}^{6} a(n,\ell_i,m)\right]$$
$$\times G_n(\ell_1,\ell_5,\ell_6,m)\, G_n(\ell_2,\ell_4,\ell_6,m)\, G_n(\ell_3,\ell_4,\ell_5,m) \tag{10}$$

and thus have reduced it to three elementary integrals of the type

$$G_n(j_1, j_2, j_3, m) := \int_0^\pi d\theta\ \sin^{2m+n-2}\theta \times C_{j_1}^{n/2-1}(\cos\theta)\, C_{j_2-m}^{m+n/2-1}(\cos\theta)\, C_{j_3-m}^{m+n/2-1}(\cos\theta). \tag{11}$$

This integral is a special case of a class of integrals already studied in [1], cf (41)[1], where we have been able to represent such integrals by three finite sums. However, because of its special form we have decided to evaluate (11) in a different way. In doing so we first recall the recurrence relation [10] for the Gegenbauer polynomials,

$$C_j^\lambda(x) = \frac{\lambda}{j+\lambda}\left[C_j^{\lambda+1}(x) - C_{j-2}^{\lambda+1}(x)\right] \tag{12}$$

which is also valid for $j = 0, 1$ if we use the convention that Gegenbauer polynomials with a 'negative degree' (the lower index) vanish identically. Iterating this recurrence relation m times we find

$$C_j^{n/2-1}(\cos\theta) = \sum_{k=0}^{\min\{m,[j/2]\}} (-1)^k\, b(n,j,k,m)\, C_{j-2k}^{m+n/2-1}(\cos\theta) \tag{13}$$

where we have introduced

$$b(n,j,k,m) := \frac{m!\,\Gamma(m+(n-2)/2)\,\Gamma(j-k+(n-2)/2)}{k!\,(m-k)!\,\Gamma((n-2)/2)\,\Gamma(j+m-k+n/2)}\,(j+m-2k+(n-2)/2). \tag{14}$$

Now replacing the first Gegenbauer polynomial in (11) with the help of (13) we realize that the integral (11) represents in essence a $3j$-symbol of the group $SO(2m+n)$, cf (21)[1]:

$$\begin{aligned} G_n(j_1, j_2, j_3, m) = {} & \frac{\sqrt{\pi}\,\Gamma(m+(n-1)/2)}{\Gamma(m+n/2)\,[(2m+n-3)!]^3} \sum_{k=0}^{\min\{m,[j_1/2]\}} (-1)^k\, b(n, j_1, k, m) \\ & \times \frac{(j_1-2k+2m+n-3)!\,(j_2+m+n-3)!\,(j_3+m+n-3)!}{(j_1-2k)!\,(j_2-m)!\,(j_3-m)!} \\ & \times \begin{pmatrix} j_1-2k & j_2-m & j_3-m \\ 0 & 0 & 0 \end{pmatrix}^2_{(n+2m)}. \end{aligned} \tag{15}$$

Thus we have succeeded in expressing the integral (11) by a single finite sum and in turn found a rather simple expression for the integral (1) in terms of four finite sums and $3j$-symbols of $SO(2m+n)$ with $m = 0, 1, \ldots, \min\{\ell_4, \ell_5, \ell_6\}$. Expression (10) together with (6), (14) and (15) thus provides us with an elementary formula for $I_n(\ell_1, \ell_2, \ell_3|\ell_4, \ell_5, \ell_6)$, which is fairly simple and can easily be evaluated using, for example, some computer-algebra program like *Mathematica*†. We also note that in our result gamma functions with a half-integer argument always occur in terms of a quotient and therefore I_n is, for given integer ℓ's and n, a rational number.

3. Discussion

In this section we will briefly discuss the symmetry properties of $I_n(\ell_1, \ell_2, \ell_3|\ell_4, \ell_5, \ell_6)$, the corresponding $6j$-symbol and its relation to the result of Ališauskas [3]. First we note that the

† A *Mathematica* package, which implements the results of this paper can be obtained from the authors at `http://theorie1.physik.uni-erlangen.de/hormess`.

Table 1. Explicit expressions for the non-vanishing integrals $I_n(\ell_1, \ell_2, \ell_3|\ell_4, \ell_5, \ell_6)$ defined in (1) and the corresponding quantity $c^{(\alpha)}_{\ell_1\ell_2\ell_3\ell_4\ell_5\ell_6}$ defined in (17).

$(\ell_1\ell_2\ell_3\ell_4\ell_5\ell_6)$	$I_n(\ell_1, \ell_2, \ell_3\|\ell_4, \ell_5, \ell_6)$	$c^{(\alpha)}_{\ell_1\ell_2\ell_3\ell_4\ell_5\ell_6}$
(1 1 2 1 1 2)	$\frac{4(n-2)}{(n-1)n^3(n+2)^3}$	$\frac{(n-2)(n-1)n}{n+2}$
(1 1 2 1 3 2)	$\frac{24}{(n-1)^2n(n+2)^3(n+4)}$	$\frac{(n-1)n^3}{n+2}$
(1 1 2 2 2 1)	$\frac{8(n-2)}{(n-1)^2n^2(n+2)^3}$	$(n-2)(n-1)n$
(1 1 2 2 2 3)	$\frac{48(n-2)}{(n-1)^3n(n+2)^3(n+4)^2}$	$\frac{(n-2)(n-1)n^2}{n+4}$
(1 1 2 2 4 3)	$\frac{288}{(n-1)^3n(n+2)^2(n+4)^2(n+6)}$	$\frac{(n-1)n^3(n+1)}{2(n+4)}$
(1 1 2 3 3 2)	$\frac{72(n-2)(n+1)}{(n-1)^3n^2(n+2)^3(n+4)^2}$	$\frac{(n-2)(n-1)n^2(n+1)}{2(n+2)}$
(1 1 2 3 3 4)	$\frac{864(n-2)}{(n-1)^3n^3(n+2)(n+4)^2(n+6)^2}$	$\frac{(n-2)(n-1)n^2(n+1)}{2(n+6)}$
(1 1 2 4 4 3)	$\frac{1152(n-2)}{(n-1)^3n^3(n+1)(n+4)^2(n+6)^2}$	$\frac{(n-2)(n-1)n^2(n+1)(n+2)}{6(n+4)}$
(1 2 3 1 2 3)	$\frac{72(n-2)}{(n-1)^3n(n+2)^3(n+4)^3}$	$\frac{(n-2)(n-1)n^3}{2(n+2)(n+4)}$
(1 2 3 1 4 3)	$\frac{864}{(n-1)^3n^2(n+2)(n+4)^3(n+6)}$	$\frac{(n-1)n^3(n+1)}{2(n+4)}$
(1 2 3 2 3 2)	$\frac{288(n-2)(n+1)}{(n-1)^4n(n+2)^3(n+4)^3}$	$\frac{(n-2)(n-1)n^2(n+1)}{n+4}$
(1 2 3 2 3 4)	$\frac{1728(n-2)}{(n-1)^4n(n+2)^2(n+4)^3(n+6)^2}$	$\frac{(n-2)(n-1)n^3(n+1)}{2(n+4)(n+6)}$
(1 2 3 3 2 3)	$\frac{864(n-2)(n+1)}{(n-1)^4n(n+2)^3(n+4)^3(n+6)}$	$\frac{(n-2)(n-1)n^3(n+1)}{(n+2)(n+6)}$
(1 2 3 3 4 3)	$\frac{5184(n-2)}{(n-1)^4n^2(n+2)(n+4)^3(n+6)^2}$	$\frac{(n-2)(n-1)n^3(n+1)}{2(n+6)}$
(1 2 3 4 3 2)	$\frac{864(n-2)}{(n-1)^4n(n+2)^2(n+4)^3(n+6)}$	$\frac{(n-2)(n-1)n^3(n+1)}{4(n+4)}$
(1 2 3 4 3 4)	$\frac{20736(n-2)}{(n-1)^4n^2(n+1)(n+4)^3(n+6)^2(n+8)}$	$\frac{(n-2)(n-1)n^3(n+1)(n+2)}{2(n+4)(n+8)}$
(1 3 4 1 3 4)	$\frac{3456(n-2)}{(n-1)^3n^3(n+1)(n+4)^3(n+6)^3}$	$\frac{(n-2)(n-1)n^3(n+1)}{6(n+4)(n+6)}$
(1 3 4 2 4 3)	$\frac{20\,736(n-2)(n+2)}{(n-1)^4n^3(n+1)(n+4)^3(n+6)^3}$	$\frac{(n-2)(n-1)n^2(n+1)(n+2)^2}{2(n+4)(n+6)}$
(1 3 4 3 3 4)	$\frac{62\,208(n-2)}{(n-1)^4n^2(n+1)(n+4)^3(n+6)^3(n+8)}$	$\frac{(n-2)(n-1)n^4(n+1)}{2(n+6)(n+8)}$
(1 3 4 4 4 3)	$\frac{124\,416(n-2)(n+3)}{(n-1)^4n^2(n+1)^2(n+4)^3(n+6)^3(n+8)}$	$\frac{(n-2)(n-1)n^4(n+1)(n+3)}{4(n+4)(n+8)}$
(2 2 2 2 2 2)	$\frac{64(n-2)(n^2+4n-24)}{(n-1)^5(n+2)^3(n+4)^3}$	$\frac{(n-2)(n-1)(n+2)^3(n^2+4n-24)}{(n+4)^3}$
(2 2 2 2 2 4)	$\frac{768(n-2)n}{(n-1)^5(n+2)^3(n+4)^3(n+6)}$	$\frac{(n-2)(n-1)n^2(n+1)(n+2)^2}{(n+4)^3}$

Table 1. Continued.

$(\ell_1\ell_2\ell_3\ell_4\ell_5\ell_6)$	$I_n(\ell_1,\ell_2,\ell_3\|\ell_4,\ell_5,\ell_6)$
(222244)	$\frac{4608(n-2)}{(n-1)^5(n+1)(n+2)(n+4)^3(n+6)^2}$
(222333)	$\frac{864(n-2)(n+1)(n^3+8n^2-28n-48)}{(n-1)^5n^2(n+2)^3(n+4)^3(n+6)^2}$
(222444)	$\frac{18\,432(n-2)(n^3+12n^2-24n-128)}{(n-1)^5n^2(n+1)^2(n+4)^3(n+6)^2(n+8)^2}$
(224224)	$\frac{2304(n-2)n^2}{(n-1)^5(n+1)(n+2)^3(n+4)^3(n+6)^3}$
(224244)	$\frac{55\,296(n-2)}{(n-1)^5(n+1)^2(n+4)^3(n+6)^3(n+8)}$
(224333)	$\frac{20\,736(n-2)}{(n-1)^5(n+2)^2(n+4)^3(n+6)^3}$
(224442)	$\frac{13\,824(n-2)n(n+3)}{(n-1)^5(n+1)^2(n+2)^2(n+4)^3(n+6)^3}$
(224444)	$\frac{663\,552(n-2)(n+3)}{(n-1)^5(n+1)^3(n+4)^3(n+6)^3(n+8)^2}$
(233233)	$\frac{2592(n-2)(n+1)(2n^4+17n^3-14n^2-84n-72)}{(n-1)^5n^3(n+2)^3(n+4)^3(n+6)^3}$
(233344)	$\frac{124\,416(n-2)(n^3+10n^2-20n-48)}{(n-1)^5n^3(n+1)(n+4)^3(n+6)^3(n+8)^2}$
(233433)	$\frac{15\,552(n-2)(n^3+11n^2-48n-36)}{(n-1)^5n^3(n+2)(n+4)^3(n+6)^3(n+8)}$
(244244)	$\frac{221\,184(n-2)(n+2)(3n^4+40n^3+72n^2-192n-512)}{(n-1)^5n^3(n+1)^3(n+4)^3(n+6)^3(n+8)^3}$
(244444)	$\frac{3\,981\,312(n-2)(n+2)(n+3)(n^3+14n^2-16n-128)}{(n-1)^5n^2(n+1)^4(n+4)^3(n+6)^3(n+8)^3(n+10)}$
(334334)	$\frac{186\,624(n-2)(4n^2+37n-50)}{(n-1)^5n^2(n+1)(n+4)^3(n+6)^3(n+8)^3}$
(334443)	$\frac{373\,248(n-4)(n-2)(n+3)(n+20)}{(n-1)^5n^2(n+1)^2(n+4)^3(n+6)^3(n+8)^3}$
(444444)	$\frac{11\,943\,936(n-2)(n+3)(n^6+43n^5+400n^4-212n^3-6752n^2-5888n+15\,360)}{(n-1)^5n^2(n+1)^5(n+4)^3(n+6)^3(n+8)^3(n+10)^3}$

$3j$-symbol (3) is obviously invariant under any permutation of the ℓ's. In addition, we note that because of the first $3j$-symbol appearing on the right-hand side of (2), the phase factor in front of it may be replaced by $(-1)^{\ell_1+\ell_2+\ell_3+\ell_4+\ell_5+\ell_6}$ as $\ell_1+\ell_2+\ell_3$ is required to be an even integer. As a consequence $I_n(\ell_1,\ell_2,\ell_3|\ell_4,\ell_5,\ell_6)$ and the $6j$-symbol have identical symmetry properties. Using the invariance property of the Haar measure in (1) one easily verifies that

$$\begin{Bmatrix} \ell_1 & \ell_2 & \ell_3 \\ \ell_4 & \ell_5 & \ell_6 \end{Bmatrix}_{(n)} = \begin{Bmatrix} \ell_2 & \ell_3 & \ell_1 \\ \ell_5 & \ell_6 & \ell_4 \end{Bmatrix}_{(n)} = \begin{Bmatrix} \ell_3 & \ell_1 & \ell_2 \\ \ell_6 & \ell_4 & \ell_5 \end{Bmatrix}_{(n)} = \begin{Bmatrix} \ell_2 & \ell_1 & \ell_3 \\ \ell_5 & \ell_4 & \ell_6 \end{Bmatrix}_{(n)}$$
$$= \begin{Bmatrix} \ell_1 & \ell_3 & \ell_2 \\ \ell_4 & \ell_6 & \ell_5 \end{Bmatrix}_{(n)} = \begin{Bmatrix} \ell_3 & \ell_2 & \ell_1 \\ \ell_6 & \ell_5 & \ell_4 \end{Bmatrix}_{(n)} = \begin{Bmatrix} \ell_1 & \ell_5 & \ell_6 \\ \ell_4 & \ell_2 & \ell_3 \end{Bmatrix}_{(n)}. \qquad (16)$$

These are indeed the well known [11] symmetries of the $6j$-symbols for the group $SO(3)$, which are thus shown to be valid for all $n \geqslant 3$ if class-one representations are considered

Table 1. Continued.

$(\ell_1\ell_2\ell_3\ell_4\ell_5\ell_6)$	$c^{(\alpha)}_{\ell_1\ell_2\ell_3\ell_4\ell_5\ell_6}$
(222244)	$\frac{(n-2)(n-1)n^2(n+1)(n+2)^3}{2(n+4)^3}$
(222333)	$\frac{(n-2)(n-1)n(n+1)(n^3+8n^2-28n-48)}{2(n+6)^2}$
(222444)	$\frac{(n-2)(n-1)n(n+1)(n+2)^3(n+6)(n^3+12n^2-24n-128)}{6(n+4)^3(n+8)^2}$
(224224)	$\frac{(n-2)(n-1)n^4(n+1)(n+2)}{4(n+4)^3(n+6)}$
(224244)	$\frac{(n-2)(n-1)n^3(n+1)(n+2)^3}{2(n+4)^3(n+8)}$
(224333)	$\frac{(n-2)(n-1)n^4(n+1)}{(n+6)^2}$
(224442)	$\frac{(n-2)(n-1)n^4(n+1)(n+2)(n+3)}{8(n+4)^3}$
(224444)	$\frac{(n-2)(n-1)n^4(n+1)(n+2)^2(n+3)(n+6)}{2(n+4)^3(n+8)^2}$
(233233)	$\frac{(n-2)(n-1)n(n+1)(n+4)(2n^4+17n^3-14n^2-84n-72)}{2(n+2)(n+6)^3}$
(233344)	$\frac{(n-2)(n-1)n^2(n+1)(n+2)(n^3+10n^2-20n-48)}{2(n+6)(n+8)^2}$
(233433)	$\frac{(n-2)(n-1)n^2(n+1)(n+4)(n^3+11n^2-48n-36)}{4(n+6)^2(n+8)}$
(244244)	$\frac{(n-2)(n-1)n(n+1)(n+2)^3(n+6)(3n^4+40n^3+72n^2-192n-512)}{6(n+4)^3(n+8)^3}$
(244444)	$\frac{(n-2)(n-1)n^3(n+1)(n+2)^2(n+3)(n+6)^2(n^3+14n^2-16n-128)}{4(n+4)^3(n+8)^3(n+10)}$
(334334)	$\frac{(n-2)(n-1)n^4(n+1)(n+4)(4n^2+37n-50)}{4(n+6)(n+8)^3}$
(334443)	$\frac{(n-4)(n-2)(n-1)n^4(n+1)(n+3)(n+20)}{8(n+8)^3}$
(444444)	$\frac{(n-2)(n-1)n^4(n+1)(n+3)(n+6)^3(n^6+43n^5+400n^4-212n^3-6752n^2-5888n+15\,360)}{16(n+4)^3(n+8)^3(n+10)^3}$

only. The additional Regge symmetry [12] known for the case $n = 3$ cannot be verified by these methods and thus it is not clear whether it holds for arbitrary $n > 3$. In combination with these symmetry properties table 1 presents for all $\ell_i \in \{1, 2, 3, 4\}$ explicit values for $I_n(\ell_1, \ell_2, \ell_3|\ell_4, \ell_5, \ell_6)$ and

$$c^{(\alpha)}_{\ell_1\ell_2\ell_3\ell_4\ell_5\ell_6} := d_{\ell_1}\, d_{\ell_2}\, d_{\ell_3}\, d_{\ell_4}\, d_{\ell_5}\, d_{\ell_6}\, I_n(\ell_1, \ell_2, \ell_3|\ell_4, \ell_5, \ell_6) \tag{17}$$

where $d_\ell := (2\ell+n-2)(\ell+n-3)!/[\ell!(n-2)!]$ denotes the dimension of the ℓth representation. Note that the quantity (17) is the actual contribution of the α-topology to the high-temperature expansion of $SO(n)$-symmetric lattice models [5]. Thus with the tabulated quantities (17) one can derive high-temperature expansions for $SO(n)$-symmetric lattice models to rather high order in the inverse temperature. For example, with only a few of the tabulated values of (17) one can find all expansion coefficients up to order ten for the specific heat [13] of a mixed isovector–isotensor model, which recently has attracted much attention [9, 14].

Finally, we would like to comment on the relation of our result with that of Ališauskas [3] on the $6j$-symbol. First we recall that with our explicit result (10) for I_n we have, with the help of (2), a similar representation for the $6j$-symbol, at least for those cases where the additional $3j$-symbols appearing on the right-hand side of (2) do not vanish. Here the result has been derived via explicit group integration, whereas Ališauskas [3] uses a series representation of the $6j$-symbol in terms of isoscalar factors. Indeed, this representation (equation (5.1) in [3]) is very much similar in form to our result (10) for I_n. Note that the quantity G_n defined in (11) is, in fact, closely related to an isoscalar factor of $SO(n)$, cf $(41)^1$–$(44)^1$. In addition to that, Ališauskas [3] was also able to show that these isoscalars may be expressed in terms of (generalized) $6j$-coefficients of $SU(2)$ which further allowed him to simplify his series representation to three finite sums, see (5.7) in [3] which is valid for $n \geqslant 5$. In contrast to this, we have considered not the $6j$-symbol itself but the group integral I_n and represented it by four finite sums. As long as the involved representation labels ℓ are small enough, which is actually the case for a high-temperature expansion, this does not cause any disadvantage. The advantage of considering $I_n(\ell_1, \ell_2, \ell_3|\ell_4, \ell_5, \ell_6)$, respectively, $c^{(\alpha)}_{\ell_1\ell_2\ell_3\ell_4\ell_5\ell_6}$, is that the resulting expressions (see table 1) are valid for all $n \geqslant 2$ and thus allow for a general discussion of the high-temperature expansion of $SO(n)$-symmetric lattice models including the important XY-model ($n = 2$) and Heisenberg model ($n = 3$).

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